ON THE FREE CONVOLUTION WITH A FREE MULTIPLICATIVE ANALOGUE OF THE NORMAL DISTRIBUTION

PING ZHONG

ABSTRACT. We obtain a formula for the density of the free convolution of an arbitrary probability measure on the unit circle of $\mathbb{C}$ with the free multiplicative analogues of the normal distribution on the unit circle. This description relies on a characterization of the image of the unit disc under the subordination function, which also allows us to prove some regularity properties of the measures obtained in this way. As an application, we give a new proof for Biane's classic result on the densities of the free multiplicative analogue of the normal distributions. We obtain analogue results for probability measures on $\mathbb{R}^+$. Finally, we describe the density of the free multiplicative analogue of the normal distributions as an example and prove unimodality and some symmetry properties of these measures.

1. INTRODUCTION

Given probability measures μ, ν on either the unit circle $\mathbb{T} \subset \mathbb{C}$, or the positive half line $\mathbb{R}^+ = [0, +\infty)$, we denote by $\mu \boxtimes \nu$ their free multiplicative convolution. Multiplicative free Brownian motion is a noncommutative stochastic process with stationary increments, which have the same distribution as the free multiplicative analogue of the normal distributions. For $t > 0$, we denote by λ_t the free multiplicative analogue of the normal distribution on $\mathbb{T}$, with Σ-transform $\Sigma_{\lambda_t}(z) = e^{\frac{t}{2}\frac{1+z}{1-z}}$. Given a probability measure μ on $\mathbb{T}$, the free Brownian motion with initial distribution μ is the noncommutative stochastic process with distribution $\{\mu \boxtimes \lambda_t : t \geq 0\}$. For $t > 0$, we denote by σ_t the free multiplicative analogue of the normal distribution of the positive half line with Σ-transform $\Sigma_{\sigma_t}(z) = e^{\frac{t}{2}\frac{1+z}{z-1}}$. Given a probability measure ν on the positive half line, similarly, the free Brownian motion with initial distribution ν is the noncommutative stochastic process with distribution $\{\nu \boxtimes \sigma_t : t \geq 0\}$.

Biane studied multiplicative free Brownian motion in his pioneering papers [6, 8]. For example, in [6] he proved that the free Brownian motions can be approximated by matrix valued Brownian motions, and calculated the moments of λ_t and σ_t. In [8] he gave a description of the density of λ_t and σ_t (see also [11, 15] for different approaches). In this article we use analytic methods to study the densities of $\mu \boxtimes \lambda_t$ and $\nu \boxtimes \sigma_t$ for an arbitrary measure μ on $\mathbb{T}$ and an arbitrary measure ν on the positive half line.

We denote by $\mathcal{M}_{\mathbb{T}}$ the set of probability measures on $\mathbb{T}$, and by $\mathcal{ID}(\boxtimes, \mathbb{T})$ the set of probability measures on $\mathbb{T}$ which are infinitely divisible with respect to $\boxtimes$. We also denote by $\mathcal{M}_{\mathbb{R}^+}$ the set of probability measures on $\mathbb{R}^+$, and by $\mathcal{ID}(\boxtimes, \mathbb{R}^+)$ the set of probability measures on $\mathbb{R}^+$ which are infinitely divisible with respect to $\boxtimes$.

Denote by P_0 the Haar measure on $\mathbb{T}$, we have that $P_0 \boxtimes \lambda_t = P_0$, and therefore we restrict ourselves to measures $\mu \in \mathcal{M}_{\mathbb{T}} \backslash \{P_0\}$. Given $\mu \in \mathcal{M}_{\mathbb{T}} \backslash \{P_0\}$, we set $\mu_t = \mu \boxtimes \lambda_t$. The

method which we use to study the density of μ_t relies on a characterization of the image $\Omega_{t,\mu}$ of the unit disc under the subordination map of η_{μ_t} with respect to η_μ. Due to a result in [2], $\Omega_{t,\mu}$ is simply connected and its boundary is a simple closed curve. We prove that the line segment $\{re^{i\theta} : 0 < r \leq 1\}$ only intersects $\partial\Omega_{t,\mu}$ at one point for any $\theta \in [0, 2\pi)$. We show that the measure μ_t is absolutely continuous with respect to P_0, and its density is analytic whenever it is positive. In addition, we obtain a formula to describe the density of μ_t, and we prove that the number of connected components of the support of μ_t is a non-increasing function of t. The case when μ is the Dirac measure at 1 yields a new proof for Biane's result on the description of the density of λ_t in [8].

We prove similar results for the positive half line. Given $\nu \in \mathcal{M}_{\mathbb{R}^+}$ which is not concentrated at 0, set $\nu_t = \nu \boxtimes \sigma_t$. Denote by $\Gamma_{t,\nu}$ the image of the upper half plane under the subordination map of η_{ν_t} with respect to η_ν. We show that for every $r > 0$ the semicircle $\{re^{i\theta} : 0 \leq \theta \leq \pi\}$ intersects $\partial\Gamma_{t,\nu}$ at exactly two points, including one point on the negative half line. We give a formula for the density of ν_t, and prove, among other results, that the number of connected components of the support of ν_t is a non-increasing function of t. When ν is the Dirac measure at 1, we obtain a new proof for Biane's [8] description of the density of σ_t. In addition, we prove some symmetry properties of the measure σ_t.

Our results are multiplicative analogues of results from [7], where the additive free convolution with a semi-circular distribution was studied. We were also influenced by the nice exposition in [10]. Our reference for free probability theory is the classical book [14] by Voiculescu, Dykema and Nica, and our reference for the properties of certain subordination functions is Belinschi and Bercovici [2].

In a joint work with H.-W. Huang in [12], we extend our method to study infinitely divisible measures and regularity properties of measures in partially defined semigroups relative to multiplicative free convolution.

This article is organized as follows. After this introductory section, we review some preliminaries regarding multiplicative free convolution in Section 2. In Section 3, we study the distribution of multiplicative free Brownian motion on $\mathbb{T}$. In Section 4, we study the distribution of multiplicative free Brownian motion on $\mathbb{R}^+$.

2. PRELIMINARIES

2.1. Multiplicative free convolution of measures on $\mathbb{T}$.

For $\mu \in \mathcal{M}_\mathbb{T}$, we define

$$\psi_\mu(z) = \int_\mathbb{T} \frac{tz}{1 - tz} d\mu(t),\ z \in \mathbb{D}$$

where $\mathbb{D}$ denotes the open unit disc, and set $\eta_\mu(z) = \psi_\mu(z)/(1 + \psi_\mu(z))$. Then $\eta_\mu(0) = 0$ and $\eta'_\mu(0)$ equals the first moment of μ. If μ has nonzero first moment, we define its Σ-transform by

$$\Sigma_\mu(z) = \frac{\eta_\mu^{-1}(z)}{z}$$

for z in a neighborhood of zero. The binary operation $\boxtimes$ on $\mathcal{M}_\mathbb{T}$, introduced in [13, 4], represents the distribution of the product of free unitary random variables. If $\mu, \nu \in \mathcal{M}_\mathbb{T}$ both have nonzero first moment, then $\mu \boxtimes \nu$ is uniquely determined by

$$\Sigma_{\mu\boxtimes\nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z)$$

for z in some domain where all three functions involved are defined. Given $\mu, \nu \in \mathcal{M}_{\mathbb{T}}$, it is proved in [9, Theorem 3.5] that there exists an analytic map $\eta : \mathbb{D} \to \mathbb{D}$, which we call the subordination function of $\eta_{\mu \boxtimes \nu}$ with respect to η_μ, such that $\eta(0) = 0$ and

$$\eta_{\mu \boxtimes \nu}(z) = \eta_\mu(\eta(z))$$

holds for all $z \in \mathbb{D}$. If both μ and ν have zero first moment, then $\mu \boxtimes \nu = P_0$, the Haar measure on $\mathbb{T}$. We are interested in the case when ν has nonzero first moment, and in particular $\nu = \lambda_t$. As it was pointed out in [15, Example 3.5], when μ has zero fist moment, the subordination function of $\eta_{\mu \boxtimes \nu}$ with respect to η_μ is generally not unique. However, if we require a subordination function satisfying one additional property, then there is a unique one, which we call the principal subordination function. The next result was first proved in [9], and then in [3] by a different method. In [15], we proved the uniqueness of the subordination functions for the case when one of the measures has zero first moment.

Theorem 2.1 ([3, 9])**.** *Given $\mu, \nu \in \mathcal{M}_{\mathbb{T}}$ such that ν has nonzero first moment, there exist two unique analytic functions $\omega_1, \omega_2 : \mathbb{D} \to \mathbb{D}$ such that*
(1) $\omega_1(0) = \omega_2(0) = 0$.
(2) $\eta_{\mu \boxtimes \nu}(z) = \eta_\mu(\omega_1(z)) = \eta_\nu(\omega_2(z))$.
(3) $\omega_1(z)\omega_2(z) = z\eta_{\mu \boxtimes \nu}(z)$ *for all* $z \in \mathbb{D}$.

Given $\mu \in \mathcal{M}_{\mathbb{T}}$, we set $\mu_t = \mu \boxtimes \lambda_t$. We also denote by η_t and ζ_t the unique subordination function satisfying the equations
(1) $\eta_t(0) = \zeta_t(0) = 0$,
(2) $\eta_{\mu_t}(z) = \eta_\mu(\eta_t(z)) = \eta_{\lambda_t}(\zeta_t(z))$ and
(3) $\eta_t(z)\zeta_t(z) = z\eta_{\mu_t}(z)$.
By a characterization of the η-transforms of measures on $\mathbb{T}$ in [2, Proposition 3.2], for any $t > 0$ there exists a unique measure $\rho_t \in \mathcal{M}_{\mathbb{T}}$ such that

$$\eta_t(z) = \eta_{\rho_t}(z) \tag{2.1}$$

holds for all $z \in \mathbb{D}$. We now recall a result in [15, Lemma 3.4 and Corollary 3.13].

Lemma 2.2 ([15])**.** *The probability measure ρ_t is $\boxtimes$-infinitely divisible and its Σ-transform is given by*

$$\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_\mu(z)) = \exp\left(\frac{t}{2}\int_{\mathbb{T}} \frac{1+\xi z}{1-\xi z} d\mu(\xi)\right). \tag{2.2}$$

Let $\Phi_{t,\mu}(z) = z\Sigma_{\rho_t}(z)$ and let $\Omega_{t,\mu} = \{z \in \mathbb{D} : |\Phi_{t,\mu}(z)| < 1\}$, then since $\rho_t \in \mathcal{ID}(\boxtimes, \mathbb{T})$, we have that $\Phi_{t,\mu}(\eta_{\rho_t}(z)) = z$ for all $z \in \mathbb{D}$. The function $\Phi_{t,\mu}$ satisfies the properties in [2, Theorem 4.4, Proposition 4.5], thus the function $\eta_t(z) = \eta_{\rho_t}(z)$ can be extended continuously to the boundary $\partial\mathbb{D}$.

Proposition 2.3 ([2])**.** (1) *The function η_t is a conformal map with image $\Omega_{t,\mu}$, and its inverse is the restriction of $\Phi_{t,\mu}$ to $\Omega_{t,\mu}$. In addition, η_t extends to be a continuous function on $\overline{\mathbb{D}}$ and η_t is one-to-one on $\overline{\mathbb{D}}$.*
(2) *$\Omega_{t,\mu}$ is a simply connected domain bounded by a simple closed curve.*
(3) *If $\xi \in \mathbb{T}$ satisfies $\eta_t(\xi) \in \mathbb{D}$, then η_t can be continued analytically to a neighborhood of ξ.*

Corollary 2.4. *The function η_μ has a continuous extension to $\overline{\Omega_{t,\mu}}$, and the function η_{μ_t} has a continuous extension to $\overline{\mathbb{D}}$.*

Proof. From Lemma 2.2 and (2.1), we see that

$$\Sigma_{\rho_t}(\eta_{\rho_t}(z)) = \Sigma_{\rho_t}(\eta_t(z)) = \Sigma_{\lambda_t}(\eta_\mu(\eta_t(z))). \tag{2.3}$$

The identity $\eta_t(z)\Sigma_{\rho_t}(\eta_{\rho_t}(z)) = \Phi_{t,\mu}(\eta_{\rho_t}(z)) = z$ and (2.3) imply that

$$\frac{z}{\eta_t(z)} = \exp\left(\frac{t}{2}\frac{1+\eta_\mu(\eta_t(z))}{1-\eta_\mu(\eta_t(z))}\right) \tag{2.4}$$

holds for $z \in \mathbb{D}$. The identity $\Phi_{t,\mu}(\eta_{\rho_t}(z)) = z$ also implies that the function η_{ρ_t} is never zero in $\mathbb{D}\backslash\{0\}$, and thus $z/\eta_t(z) = z/\eta_{\rho_t}(z)$ is never zero on $\mathbb{D}$. By taking logarithms in both sides of (2.4), we see that η_t extends continuously to $\partial\Omega_{t,\mu}$ and $\eta_t(\overline{\mathbb{D}}) = \overline{\Omega_{t,\mu}}$. The conclusion of the corollary follows. ☐

We close this section with a formula which allows us to recover μ from its η-transform. For $\mu \in \mathcal{M}_\mathbb{T}$, we have that

$$\frac{1}{2\pi}\left(\frac{1+\eta_\mu(z)}{1-\eta_\mu(z)}\right) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(e^{-i\theta}),\ z \in \mathbb{D}. \tag{2.5}$$

The real part of this function is the Poisson integral of the measure $d\mu(e^{-i\theta})$, and then μ can be recovered from (2.5) by the Stieltjes inversion formula. The functions

$$\frac{1}{2\pi}\Re\left(\frac{1+\eta_\mu(re^{i\theta})}{1-\eta_\mu(re^{i\theta})}\right) = \frac{1}{2\pi}\frac{1-|\eta_\mu(re^{i\theta})|^2}{|1-\eta_\mu(re^{i\theta})|^2} \tag{2.6}$$

converge to the density of $d\mu(e^{-i\theta})$ a.e. relative to Lebesgue measure $d\theta$, and they converge to infinity a.e. relative to the singular part of this measure.

2.2. **Multiplicative free convolution of measures on $\mathbb{R}^+$.** Given $\mu \in \mathcal{M}_{\mathbb{R}^+}$, we define

$$\psi_\mu(z) = \int_0^{+\infty}\frac{tz}{1-tz}d\mu(t),\ z \in \mathbb{C}\backslash\mathbb{R}^+$$

and set $\eta_\mu(z) = \psi_\mu(z)/(1+\psi_\mu(z))$. The case when $\mu = \delta_0$, the Dirac measure at 0, is trivial, we thus set $\mathcal{M}^*_{\mathbb{R}^+} = \mathcal{M}_{\mathbb{R}^+}\backslash\{\delta_0\}$. Given $\mu \in \mathcal{M}^*_{\mathbb{R}^+}$, it is shown in [5] that the function η_μ is univalent in the left half plane $i\mathbb{C}^+$. The Σ-transform of μ is defined by

$$\Sigma_\mu(z) = \frac{\eta_\mu^{-1}(z)}{z}$$

for z in $\eta_\mu(i\mathbb{C}^+)$. Given $\mu, \nu \in \mathcal{M}^*_{\mathbb{R}^+}$, the multiplicative free convolution of μ and ν, denoted by $\mu \boxtimes \nu$, is uniquely determined by

$$\Sigma_{\mu\boxtimes\nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z) \tag{2.7}$$

in some domain where all three functions involved are defined. It is also known from [3, 9] that there exist two analytic functions, which are called subordination functions, $\omega_1, \omega_2 : \mathbb{C}\backslash\mathbb{R}^+ \to \mathbb{C}\backslash\mathbb{R}^+$ such that

(1) $\omega_j(0-) = 0$ for $j = 1, 2$.
(2) for any $\lambda \in \mathbb{C}^+$, we have $\omega_j(\overline{\lambda}) = \overline{\omega_j(\lambda)}$ for $j = 1, 2$.

(3) $\eta_{\mu\boxtimes\nu}(z) = \eta_\mu(\omega_1(z)) = \eta_\nu(\omega_2(z))$ for $z \in \mathbb{C}\backslash\mathbb{R}^+$.

Fix $\nu \in \mathcal{M}^*_{\mathbb{R}^+}$ and let ω_t be the subordination function of $\eta_{\nu\boxtimes\sigma_t}$ with respect to η_ν, that is $\eta_{\nu\boxtimes\sigma_t}(z) = \eta_\nu(\omega_t(z))$. Due to [5, Proposition 6.1], there exists a measure $\tau_t \in \mathcal{M}^*_{\mathbb{R}^+}$, such that $\omega_t(z) = \eta_{\tau_t}(z)$ for $z \in \mathbb{C}\backslash\mathbb{R}^+$. We then recall [15, Proposition 4.3] as follows.

Proposition 2.5. *The measure τ_t is $\boxtimes$-infinitely divisible and its Σ-transform is given by*

$$\Sigma_{\tau_t}(z) = \Sigma_{\sigma_t}(\eta_\nu(z)) = \exp\left(\frac{t}{2}\int_0^{+\infty}\frac{1+\xi z}{\xi z - 1}d\nu(\xi)\right). \tag{2.8}$$

Let $H_{t,\nu}(z) = z\Sigma_{\tau_t}(z)$, and let $\Gamma_{t,\nu}$ be the connected component of the set $\{z \in \mathbb{C}^+ : \Im(H_{t,\nu}(z)) > 0\}$ whose boundary contains the negative half line $(-\infty, 0)$. We then have that $H_{t,\nu}(\omega_t(z)) = z$ for $z \in \mathbb{C}^+$ and that $\omega_t(H_{t,\nu}(z)) = z$ for $z \in \Gamma_{t,\nu}$. The following result deduces that this definition of $\Gamma_{t,\nu}$ agrees with the one given in the introduction.

Proposition 2.6. (1) *The restriction of ω_t to $\mathbb{C}^+$ is a conformal map with image $\Gamma_{t,\nu}$, and its inverse is the restriction of $H_{t,\nu}$ to $\Gamma_{t,\nu}$. In addition, ω_t extends to be a continuous function on $\mathbb{C}^+ \cup \mathbb{R}$, and ω_t is one to one on this set.*

(2) *If $\xi \in \mathbb{R}^+$ satisfies $\Im(\omega_t(\xi)) > 0$, then ω_t can be continued analytically to a neighborhood of ξ.*

Proof. Since ω_t is the η-transform of τ_t and $\tau_t \in \mathcal{ID}(\boxtimes, \mathbb{R}^+)$, we have that $\omega_t = \eta_{\tau_{t/2}\boxtimes\tau_{t/2}}$. Then by [2, Proposition 5.2] and its proof, we see that ω_t extended continuous to $\mathbb{C}^+\cup\mathbb{R}\backslash\{0\}$, and the extended function is analytic at points $\xi \in (0, +\infty)$ at which the extended function is not real. This proves (2) and part of (1).

Recall that ω_t is the subordination function of $\eta_{\nu\boxtimes\sigma_t}$ with respect to η_ν, from a general result concerning subordination functions for multiplicative free convolution of arbitrary measures on $\mathbb{R}^+$ in [1, Remark 3.3], we deduce that ω_t extends continuously to 0 as well if ν is not a Dirac measure at one point. Note that λ_t is compactly supported, thus τ_t is compactly supported as well if ν is a Dirac measure at one point, we conclude that ω_t also extends continuously to 0 for this case. This finishes the proof. $\square$

From (2.8), we have the following useful formula.

$$\begin{aligned}\frac{z}{\omega_t(z)} &= \Sigma_{\tau_t}(\omega_t(z)) = \Sigma_{\lambda_t}(\eta_\nu(\omega_t(z))) \\ &= \exp\left(\frac{t}{2}\frac{\eta_{\nu_t}(z)+1}{\eta_{\nu_t}(z)-1}\right),\end{aligned} \tag{2.9}$$

which yields the next result.

Proposition 2.7. *The function η_ν has a continuous extension to $\overline{\Gamma_{t,\nu}}\backslash\{0\}$, and the function η_{ν_t} has a continuous extension to $\mathbb{C}^+ \cup (\mathbb{R}\backslash\{0\})$.*

3. The unit circle case

In this section, we prove some regularity properties of $\Omega_{t,\mu}$ and then prove our main results regarding distributions of multiplicative free Brownian motion on $\mathbb{T}$. We use polar

coordinates in our discussion and parametrize $\mathbb{T}$ as $\mathbb{T} = \{e^{i\theta} : -\pi \le \theta \le \pi\}$. For fixed $t > 0$, define

$$h_t(r, \theta) = 1 - \frac{t}{2}\frac{1 - r^2}{-\ln r}\int_{\mathbb{T}} \frac{1}{|1 - re^{i\theta}\xi|^2} d\mu(\xi).$$

We have

$$\begin{aligned} \ln(|\Phi_{t,\mu}(re^{i\theta})|) &= \ln r + \frac{t}{2}\int_{\mathbb{T}} \frac{1 - r^2}{|1 - re^{i\theta}\xi|^2} d\mu(\xi) \\ &= (\ln r)h_t(r, \theta). \end{aligned} \tag{3.1}$$

To study the boundary of $\Omega_{t,\mu}$, we need the following lemma.

Lemma 3.1. *Given $-1 \le y \le 1$, define a function of r by*

$$T_y(r) = \frac{1 - r^2}{-\ln r}\frac{1}{1 - 2ry + r^2}$$

on the interval $(0, 1)$, then $T_y'(r) > 0$ for all $r \in (0, 1)$.

Proof. The substitution $x = -\ln r$ implies that it suffices to prove $f'(x) < 0$ for $x \in (0, +\infty)$, where

$$f(x) = \frac{1 - e^{-2x}}{x}\frac{1}{1 - 2e^{-x}y + e^{-2x}}.$$

We have

$$f'(x) = \frac{(-e^{4x} + 4xe^{2x} + 1) + y(-2xe^{3x} + 2e^{3x} - 2xe^x - 2e^x)}{[x(e^{2x} - 2e^x y + 1)]^2}.$$

By Taylor expansion, we can check that $-2xe^{3x} + 2e^{3x} - 2xe^x - 2e^x < 0$ for all $x > 0$; thus, to prove $f'(x) < 0$, it is enough to show that

$$2xe^{3x} - 2e^{3x} + 2xe^x + 2e^x < e^{4x} - 4xe^{2x} - 1. \tag{3.2}$$

It is easy to check that (3.2) holds for all $x \in (0, +\infty)$ by calculating the Taylor expansions in both sides of (3.2). $\square$

Lemma 3.1 implies that the function $h_t(r, \theta)$ is a decreasing function of r on (0,1) for fixed θ. Define $h_t : [-\pi, \pi] \to \mathbb{R} \cup \{-\infty\}$ as follows:

$$\begin{aligned} h_t(\theta) &= \lim_{r \to 1^-} h_t(r, \theta) \\ &= 1 - t\int_{-\pi}^{\pi} \frac{1}{|1 - e^{i(\theta+x)}|^2} d\mu(e^{ix}) \\ &= 1 - t\int_{-\pi}^{\pi} \frac{1}{2 - 2\cos(\theta + x)} d\mu(e^{ix}). \end{aligned}$$

We now let

$$\begin{aligned} U_{t,\mu} &= \{-\pi \le \theta \le \pi : h_t(\theta) < 0\} \\ &= \left\{-\pi \le \theta \le \pi : \int_{-\pi}^{\pi} \frac{1}{|1 - e^{i(\theta+x)}|^2} d\mu(e^{ix}) > \frac{1}{t}\right\} \end{aligned}$$

and $U^c_{t,\mu} = [-\pi, \pi]\backslash U_{t,\mu}$. We also define a function $v_t : [-\pi, \pi] \to [0, 1)$ as

$$
\begin{aligned}
v_t(\theta) &= \inf\{0 \le r < 1 : h_t(r,\theta) \ge 0\} \\
&= \inf\left\{0 \le r < 1 : \frac{1-r^2}{-2\ln r}\int_{-\pi}^{\pi} \frac{1}{|1 - re^{i(\theta+x)}|^2} d\mu(e^{ix}) \le \frac{1}{t}\right\}.
\end{aligned} \tag{3.3}
$$

The next result regarding regularity of $\Omega_{t,\mu}$ is fundamental to our discussion. Note that this type of result was proved in [15] in the study of the density of λ_t.

Theorem 3.2. *The sets $\Omega_{t,\mu}$ and $\partial\Omega_{t,\mu}$ can be characterized by the functions $h_t(r,\theta)$ and $v_t(\theta)$ as follows.*

(1) *If $z = r_0 e^{i\theta_0} \in \Omega_{t,\mu}$, then the entire segment $\{rz : 0 \le r < 1\}$ is contained in $\Omega_{t,\mu}$.*
(2) *$\Omega_{t,\mu} = \{re^{i\theta} : 0 < r < v_t(\theta)\}$.*
(3) *$\partial\Omega_{t,\mu} \cap \mathbb{D} = \{re^{i\theta} : \theta \in U_{t,\mu}, \text{and } h_t(r,\theta) = 0\}$.*

Proof. From Proposition 2.3 and (3.1), we see that

$$
\begin{aligned}
\Omega_{t,\mu} \cap \mathbb{D} &= \{z \in \mathbb{D} : \ln(|\Phi_{t,\mu}(z)|) < 0\} \\
&= \{re^{i\theta} : 0 < r < 1, -\pi \le \theta \le \pi \,\text{and}\, h_t(r,\theta) > 0\}.
\end{aligned}
$$

Since $h_{t,\theta}(r) := h_t(r,\theta)$ is a decreasing function of r, we then deduce the following assertions:

(i) If $\theta \in U_{t,\mu}$, then $\{re^{i\theta} : 0 \le r < v_t(\theta)\} \subset \Omega_{t,\mu}$ and $v_t(\theta)e^{i\theta} \in \partial\Omega_{t,\mu}$.
(ii) If $\theta \in U^c_{t,\mu}$, then $\{re^{i\theta} : 0 \le r < 1\} \subset \Omega_{t,\mu}$ and $e^{i\theta} \in \partial\Omega_{t,\mu}$.

We thus proved (1) and (3). Notice that for $\theta \in U^c_{t,\mu}$, we have that $v_t(\theta) = 1$, therefore (i) and (ii) also imply (2). ☐

Lemma 3.3. *The support of μ is contained in the closure of the set $\widetilde{U_{t,\mu}}$ which is defined by*

$$
\widetilde{U_{t,\mu}} = \left\{e^{-i\theta} : \theta \in U_{t,\mu}\right\}.
$$

Proof. Let $\theta_0 \in [-\pi, \pi]\backslash\overline{U_{t,\mu}}$, where $\overline{U_{t,\mu}}$ is the closure of $U_{t,\mu}$. Without loss of generality, we may assume that θ_0 is different from π and $-\pi$, then there is some $\epsilon > 0$, such that

$$
[\theta_0 - \epsilon, \theta_0 + \epsilon] \subset (-\pi, \pi)\backslash\overline{U_{t,\mu}}.
$$

For any integer $n \ge 2$, we define $\alpha_k = \theta_0 - \epsilon + 2k\epsilon/n$ for all $0 \le k \le n$, then the sets $[\alpha_k, \alpha_{k+1}]$ are contained in $(-\pi, \pi)\backslash\overline{U_{t,\mu}}$. Given $\theta \in [\alpha_k, \alpha_{k+1}]$, then we have that

$$
\begin{aligned}
\frac{1}{t} &\ge \int_{-\alpha_{k+1}}^{-\alpha_k} \frac{1}{|1 - e^{i(\theta+x)}|^2} d\mu(e^{ix}) \\
&= \int_{-\alpha_{k+1}}^{-\alpha_k} \frac{1}{|e^{i\theta} - e^{-ix}|^2} d\mu(e^{ix}) \\
&\ge \frac{\mu(\{e^{i\theta} : -\alpha_{k+1} \le \theta \le -\alpha_k\})}{|e^{-i\alpha_k} - e^{-i\alpha_{k+1}}|^2}.
\end{aligned}
$$

This implies that

$$\mu(\{e^{i\theta} : -(\theta_0+\epsilon) \le \theta \le -(\theta_0-\epsilon)\}) \le \sum_{k=1}^{n-1} \mu(\{e^{i\theta} : -\alpha_{k+1} \le \theta \le -\alpha_k\}) \le \frac{1}{t}\sum_{k=0}^{k-1} |e^{-i\alpha_k} - e^{-i\alpha_{k+1}}|^2.$$

Notice that $\sum_{k=0}^{k-1} |e^{-i\alpha_k} - e^{-i\alpha_{k+1}}|^2$ can be arbitrarily small if we choose n large enough, therefore

$$\mu(\{e^{i\theta} : -(\theta_0+\epsilon) \le \theta \le -(\theta_0-\epsilon)\}) = 0$$

and our assertion follows. □

Proposition 3.4. *For any interval $(\alpha,\beta) \subset U_{t,\mu}^c$, the function $f(\theta) = \int_{-\pi}^{\pi} \frac{1}{|1-e^{i(\theta+x)}|^2} d\mu(e^{ix})$ is strictly convex in (α,β).*

Proof. We notice that

$$\int_{-\pi}^{\pi} \frac{1}{|1-e^{i(\theta+x)}|^2} d\mu(e^{ix}) = \int_{-\pi}^{\pi} \frac{1}{|e^{ix}-e^{-i\theta}|^2} d\mu(e^{ix}) = \int_{-\pi}^{\pi} \frac{1}{2-2\cos(\theta+x)} d\mu(e^{ix}),$$

from Lemma 3.3, we deduce that f is analytic on (α,β). We calculate its second derivative

$$f''(\theta) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1-\cos(\theta+x)+\sin^2(\theta+x)}{(1-\cos(\theta+x))^3} d\mu(e^{ix}) > 0,$$

which implies the convexity of f. □

Remark 1. Proposition 3.4 implies that $\int_{-\pi}^{\pi} \frac{1}{|1-e^{i(\theta+x)}|^2} d\mu(e^{ix}) < 1/t$ for θ in the interior of $U_{t,\mu}^c$.

We are now ready to discuss the density of the measure $\mu_t = \mu \boxtimes \lambda_t$.

Proposition 3.5. *The measure μ_t is absolutely continuous with respect to Lebesgue measure. Moreover, the density is analytic whenever it is positive.*

Proof. By [15, Lemma 5.1], $\eta_{\lambda_t}(\overline{\mathbb{D}})$ does not contain 1. Since η_{μ_t} is subordinated with respect to η_{λ_t}, $\eta_{\mu_t}(\overline{\mathbb{D}})$ does not contain 1 either. Then from (2.6), we see that $\Re\left(\frac{1+\eta_{\mu_t}(e^{i\theta})}{1-\eta_{\mu_t}(e^{i\theta})}\right)$ is bounded, which implies that μ_t has no sigular part, and thus μ_t is absolutely continuous with respect to Lebesgue measure.

We rewrite (2.4) as

$$\frac{z}{\eta_t(z)} = \exp\left(\frac{t}{2}\frac{1+\eta_{\mu_t(z)}}{1-\eta_{\mu_t(z)}}\right).$$

If the density of μ_t is positive at $e^{-i\theta}$, then by (2.6), we have $|\eta_{\mu_t}(e^{i\theta})| < 1$, which implies that $|\eta_t(e^{i\theta})| < 1$. The analyticity of the density follows from part (3) of Proposition 2.3. □

Define a map $\Psi_{t,\mu} : \mathbb{T} \to \mathbb{T}$ by

$$\Psi_{t,\mu}(e^{i\theta}) = \Phi_{t,\mu}(v_t(\theta)e^{i\theta}).$$

Recall that $|\Phi_{t,\mu}(v_t(\theta)e^{i\theta})| = 1$, we calculate

$$\arg(\Phi_{t,\mu}(v_t(\theta)e^{i\theta})) = \theta + t\int_{-\pi}^{\pi} \frac{v_t(\theta)\sin(\theta + x)}{|1 - v_t(\theta)e^{i(\theta+x)}|^2} d\mu(e^{ix}).$$

Proposition 3.6. *The map $e^{i\theta} \to v_t(\theta)e^{i\theta}$ is a homeomorphism from $\mathbb{T}$ onto $\partial\Omega_{t,\mu}$; and the map $\Psi_{t,\mu}$ is a homeomorphism from $\mathbb{T}$ onto $\mathbb{T}$.*

Theorem 3.7. *The measure μ_t has a density given by*

$$p_t(\overline{\Psi_{t,\mu}(e^{i\theta})}) = -\frac{\ln v_t(\theta)}{\pi t}. \tag{3.4}$$

Proof. We prove it by a direct calculation.

$$\begin{aligned} p_t(\overline{\Psi_{t,\mu}(e^{i\theta})}) &= p_t(\overline{\Phi_{t,\mu}(v_t(\theta)e^{i\theta})}) \\ &\overset{(1)}{=} \frac{1}{2\pi} \frac{1 - |\eta_\mu(v_t(\theta)e^{i\theta})|^2}{|1 - \eta_\mu(v_t(\theta)e^{i\theta})|^2} \\ &= \frac{1}{2\pi} \Re\left(\frac{1 + \eta_\mu(v_t(\theta)e^{i\theta})}{1 - \eta_\mu(v_t(\theta)e^{i\theta})}\right) \\ &\overset{(2)}{=} \frac{1}{2\pi} \Re\left(\int_{\mathbb{T}} \frac{1 + \xi v_t(\theta)e^{i\theta}}{1 - \xi v_t(\theta)e^{i\theta}} d\mu(\xi)\right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - v_t(\theta)^2}{|1 - v_t(\theta)e^{i(\theta+x)}|^2} d\mu(e^{ix}) \\ &\overset{(3)}{=} \frac{1}{2\pi} \frac{-2\ln v_t(\theta)}{t} = \frac{-\ln v_t(\theta)}{\pi t}. \end{aligned}$$

The equality (1) is due to (2.5), the Stieltjes's inversion formula and the fact $\eta_t(\Phi_{t,\mu}(z)) = z$ for $z \in \overline{\Omega_{t,\mu}}$; the equality (2) follows from the definition of η-transform; and the equality (3) follows from the definition of $v_t(\theta)$. $\square$

Corollary 3.8. *Let A_t be the support of the measure μ_t, and let $\widetilde{A_t} = \{e^{-i\theta} : e^{i\theta} \in A_t\}$, then $\widetilde{A_t}$ equals to the image of the closure of $U_{t,\mu}$ by the homeomorphism $\Psi_{t,\mu}$. The number of connected components of the interior of the support of μ_t is a non-increasing function of t.*

Proof. The first assertion follows from Theorem 3.7 and the fact that

$$v_t(\theta) \neq 0 \Leftrightarrow \theta \in U_{t,\mu}.$$

From Theorem 3.7, to prove the second assertion, it is enough to prove that the number of connected components of $U_{t,\mu}$ is a non-increasing function of t, which is a consequence of Proposition 3.4. $\square$

Proposition 3.9. *The number $v_t(\theta)$ is bounded below by a_t, where a_t is the unique solution of the equation*

$$r\exp\left(\frac{t}{2}\frac{1+r}{1-r}\right) = 1;$$

and thus the density of μ_t is bounded above by $-\ln a_t/\pi t$. Moreover, the density of μ_t tends to $1/2\pi$ uniformly as $t \to \infty$; in particular, the support of μ_t is $\mathbb{T}$ when t is sufficiently large.

Proof. We notice that

$$\frac{1-r^2}{-2\ln r}\int_{-\pi}^{\pi}\frac{1}{|1-re^{i(\theta+x)}|^2}d\mu(e^{ix}) \le \frac{1+r}{(-2\ln r)(1-r)}. \tag{3.5}$$

By taking $y=1$ for Lemma 3.1, we see that the right hand side of (3.5) is also an increasing function of r, which yields that $v_t(\theta) \ge a_t$. The last assertion is a consequence of [15, Corollary 3.27]. □

Remark 2. From the description of the density of λ_t in Example 3.10 below, or from [15, Theorem 5.4], we see that $-\ln a_t/\pi t$ is the maximum of the density of λ_t.

As an example, we now apply Theorem 3.7 to give a description of the density of λ_t which recovers Biane's classic results regarding the density of λ_t proved in [8]. See also [11, 15] for different approaches. We provide a picture of the density function of λ_t for some values of t in the end of this paper.

Example 3.10. Let $\mu = \delta_1$, the Dirac measure at 1, we denote $\Omega_t = \Omega_{t,\delta_1}$, $\Phi_t = \Phi_{t,\delta_1}$, $\Psi_t = \Psi_{t,\delta_1}$ and $U_t = U_{t,\delta_1}$. Then we have that $\mu_t = \lambda_t$ and

$$U_t = \left\{-\pi \le \theta \le \pi : \frac{1}{|1-e^{i\theta}|^2} > \frac{1}{t}\right\} = \left\{-\pi \le \theta \le \pi : 1-\cos\theta < \frac{t}{2}\right\}$$

which yields that $U_t = \mathbb{T}$ if $t > 4$, and

$$U_t = (-\arccos(1-t/2), \arccos(1-t/2))$$

if $t \le 4$. If $\theta \in U_t$, then $v_t(\theta)$ satisfies the equation

$$\frac{1-v_t(\theta)^2}{-2\ln v_t(\theta)}\frac{1}{1-2v_t(\theta)\cos\theta + v_t(\theta)^2} = \frac{1}{t}, \tag{3.6}$$

and we have

$$\arg(\Psi_t(e^{i\theta})) = \theta + t\frac{v_t(\theta)\sin\theta}{1-2v_t(\theta)\cos\theta+v_t(\theta)^2}. \tag{3.7}$$

For $t \le 4$, let $\theta_0(t) = \arccos(1-t/2)$, then $v_t(\theta_0) = 1$, from which we deduce that

$$\arg(\Psi_t(e^{i\theta_0(t)})) = \theta_0(t) + \sin\theta_0(t). \tag{3.8}$$

From Corollary 3.8 and (3.8), we see that the support of λ_t is the set

$$\left\{e^{i\theta} : -\arccos\left(1-\frac{t}{2}\right) - \frac{1}{2}\sqrt{(4-t)t} \le \theta \le \arccos\left(1-\frac{t}{2}\right) + \frac{1}{2}\sqrt{(4-t)t}\right\}$$

if $t < 4$; and the support of λ_t is $\mathbb{T}$ if $t \ge 4$.

By Theorem 3.7 and (3.6), we obtain that the density of λ_t at $\Phi_t(v_t(\theta)e^{i\theta})$ is given by

$$\frac{1}{2\pi}\frac{1-v_t(\theta)^2}{1-2v_t(\theta)\cos\theta+v_t(\theta)} = \frac{1}{2\pi}\Re\left(\frac{1+v_t(\theta)e^{i\theta}}{1-v_t(\theta)e^{i\theta}}\right), \tag{3.9}$$

where we used the fact that $v_t(-\theta) = v_t(\theta)$ due to the identity $\Phi_t(\overline{z}) = \overline{\Phi_t(z)}$. Note that $\Phi_t(z) = z\exp(\frac{t}{2}\frac{1+z}{1-z})$, then (3.9) implies that, for $\theta \in U_t$, at the point $\omega = \Phi_t(v_t(\theta)e^{i\theta})$, the density is positive and is equal to the product of $1/2\pi$ and the real part of $k(t,\omega)$, where $k(t,\omega)$ is the only solution, with positive real part, of the equation

$$\frac{z-1}{z+1}e^{\frac{t}{2}z} = \omega.$$

Theorem 3.2 and [15, Lemma 5.1] allow us to obtain a better description for Ω_t, which implies the unimodality of λ_t as it was shown in [15, Theorem 5.4].

Proposition 3.11. *For any $t > 0$, there exists a non-decreasing function $\gamma_t : [0,\pi] \to (0,1]$ such that*

$$\partial\Omega_t = \{\gamma_t(\theta)e^{i\theta} : 0 \leq \theta \leq \pi\} \cup \{\gamma_t(\theta)e^{-i\theta} : 0 \leq \theta \leq \pi\}.$$

Proof. For given $0 < r < 1$, we notice that the map $\theta \to |\Phi_{t,\delta_1}(re^{i\theta})| = r\exp\left(\frac{t}{2}\frac{1-r^2}{1-2r\cos\theta+r^2}\right)$ is a strictly decreasing function of θ on the interval $[0,\pi]$, which implies that if $z = r_0e^{i\theta_0} \in \partial\Omega_t \cap \mathbb{D} \cap \mathbb{C}^+$ for $\theta_0 \in [0,\pi)$, then we have that

$$\{re^{i\theta} : r = r_0, \theta_0 < \theta \leq \pi\} \subset \Omega_t. \tag{3.10}$$

Note that Ω_t is symmetric with respect to x-axis, then (3.10) and Theorem 3.2 yield our assertion. □

Corollary 3.12. *The density of λ_t is symmetric with respect to x-axis; and the density of λ_t has a unique maximum at 1.*

Remark 3. For $0 < t < 4$, from (3.6), we can easily see that there exists $a(t) > 0$ such that

$$-\ln v_t(\theta) = a(t)|\theta_0(t) - \theta|^{\frac{1}{2}}(1 + o(1)) \tag{3.11}$$

in a small interval $(\theta_0(t) - \epsilon, \theta_0]$ for some $\epsilon > 0$. By taking derivative for (3.7) and noticing (3.11), we derive that

$$\frac{d}{d\theta}\arg(\Psi_t(e^{i\theta}))\bigg|_{\theta=\theta_0} > 0.$$

Set $\theta(t) = \arccos(1 - t/2) + \sqrt{(4-t)t}/2$ be one end point of the support of λ_t, then there exists $b(t) > 0$ such that

$$\frac{d\lambda_t}{d\theta}\left(e^{i\theta}\right) = b(t)|\theta - \theta(t)|^{\frac{1}{2}}(1 + o(1)) \tag{3.12}$$

in a small interval $(\theta(t) - \epsilon', \theta(t)]$ for some $\epsilon' > 0$.

By a similar calculation, we can see that there exists $b(4) > 0$ such that

$$\frac{d\lambda_4}{d\theta}\left(e^{i\theta}\right) = b(4)\,|\theta - \pi|^{\frac{1}{3}}\,(1 + o(1)) \tag{3.13}$$

in a small interval $(\pi - \epsilon', \pi]$.

We note that the function Φ_t has zeros of order one at $e^{i\theta_0(t)}$ and $e^{-i\theta_0(t)}$ for $0 < t < 4$ and Φ_4 has a zero of order two at -1. The orders in (3.12) and (3.13) are essentially due to this fact.

4. The positive half line case

In this section, we prove some useful properties of $\Gamma_{t,\nu}$, and give a description of the density of $\nu_t = \nu \boxtimes \sigma_t$.

We use polar coordinates and the parametrization $\mathbb{C}^+ = \{re^{i\theta} : 0 < r < +\infty, 0 < \theta < \pi\}$. For $z = re^{i\theta} \in \mathbb{C}^+$, we calculate

$$|H_{t,\nu}(z)| = r \exp\left(\frac{t}{2}\int_0^{+\infty} \frac{r^2\xi^2 - 1}{1 + r^2\xi^2 - 2r\xi\cos\theta} d\nu(\xi)\right)$$

and

$$\begin{aligned} \arg(H_{t,\nu}(z)) &= \theta - t\int_0^{+\infty} \frac{r\xi\sin\theta}{1 + r^2\xi^2 - 2r\xi\cos\theta} d\nu(\xi) \\ &= \theta\left[1 - \frac{t(\sin\theta)}{\theta}\int_0^{+\infty} \frac{r\xi}{1 + r^2\xi^2 - 2r\xi\cos\theta} d\nu(\xi)\right]. \end{aligned} \tag{4.1}$$

For $0 < r < +\infty$ and $0 < \theta < \pi$, we set

$$f_t(r,\theta) = 1 - \frac{t(\sin\theta)}{\theta}\int_0^{+\infty} \frac{r\xi}{1 + r^2\xi^2 - 2r\xi\cos\theta} d\nu(\xi).$$

Lemma 4.1. *For fixed $0 < r < +\infty$, the function $f_t(r,\theta)$ is an increasing function of θ on $(0,\pi)$.*

Proof. We calculate

$$\left(\frac{\sin\theta}{\theta}\right)' = \frac{\cos\theta}{\theta}(\theta - \tan\theta) < 0,$$

and

$$\frac{d}{d\theta}\left(\int_0^{+\infty} \frac{r\xi}{1 + r^2\xi^2 - 2r\xi\cos\theta} d\nu(\xi)\right) = \int_0^{+\infty} \frac{-2r^2\xi^2\sin\theta}{(1 + r^2\xi^2 - 2r\xi\cos\theta)^2} d\nu(\xi) < 0$$

for all $\theta \in (0,\pi)$ and $0 < r < \infty$. The assertion follows from these calculations. $\square$

We set $f_t(r) = \lim_{\theta\to 0^+} f_t(r,\theta)$, then we have that

$$f_t(r) = 1 - t\int_0^{+\infty} \frac{r\xi}{(1 - r\xi)^2} d\nu(\xi).$$

Let

$$\begin{aligned} V_{t,\nu} &= \{0 < r < +\infty : f_t(r) < 0\} \\ &= \left\{0 < r < +\infty : \int_0^{+\infty} \frac{r\xi}{(1 - r\xi)^2} d\nu(\xi) > \frac{1}{t}\right\}. \end{aligned}$$

We also define a function $u_t : (0,+\infty) \to [0,\pi)$ as

$$\begin{aligned} u_t(r) &= \inf\{0 \le \theta < \pi : f_t(r,\theta) \ge 0\} \\ &= \inf\left\{0 \le \theta < \pi : \frac{\sin\theta}{\theta}\int_0^{+\infty} \frac{r\xi}{1 + r^2\xi^2 - 2r\xi\cos\theta} d\nu(\xi) \le \frac{1}{t}\right\}. \end{aligned}$$

Remark 4. If $r \in V_{t,\nu}$, then $u_t(r) > 0$ and we then have

$$\frac{\sin(u_t(r))}{u_t(r)} \int_0^{+\infty} \frac{r\xi}{1 + r^2\xi^2 - 2r\xi\cos(u_t(r))} d\nu(\xi) = \frac{1}{t}.$$

The function u_t depends on ν, and we always fix an arbitrary $\nu \in \mathcal{M}_{\mathbb{R}+}^*$ in this article when we discuss u_t. Only in Proposition 4.13 and Lemma 4.14, we choose ν as $\nu = \delta_1$.

Theorem 4.2. *The sets $\Gamma_{t,\nu}, \partial\Gamma_{t,\nu}$ can be characterized by the functions $f_t(r,\theta)$ and $u_t(\theta)$ as follows.*

(1) *If $z = r_0 e^{i\theta_0} \in \Gamma_{t,\nu}$, then we have that*

$$\{z = re^{i\theta} : r = r_0, \theta_0 < \theta < \pi\} \subset \Gamma_{t,\nu}.$$

(2) $\Gamma_{t,\nu} = \{re^{i\theta} : 0 < r < +\infty, u_t(r) < \theta < \pi\}$.

(3) $\partial\Gamma_{t,\nu} \cap \mathbb{C}^+ = \{re^{i\theta} : r \in V_{t,\nu}, \textit{and}\, f_t(r,\theta) = 0\}$.

(4) *We have that*

$$\begin{aligned} \partial\Gamma_{t,\nu} \cap (0,+\infty) &= \{r : 0 < r < +\infty, u_t(r) = 0\} = (0,+\infty)\backslash V_{t,\nu} \\ &= \left\{ 0 < r < +\infty : \int_0^{+\infty} \frac{r\xi}{(1-r\xi)^2} d\nu(\xi) \leq \frac{1}{t} \right\}. \end{aligned}$$

Proof. Since $\arg(H_{t,\nu}) = \theta \cdot f_t(r,\theta)$, and the function $f_{t,r}(\theta) := f_t(r,\theta)$ is an increasing function of θ on $(0,\pi)$, notice that $\lim_{\theta\to\pi} f_{t,r}(\theta) = 1 > 0$, then from (4.1), we deduce that

$$\Gamma_{t,\nu} = \{z = re^{i\theta} : 0 < r < +\infty, f_t(r,\theta) > 0\},$$

which yields (1), (2) and (3). Part (4) follows from the definition of u_t and (3). $\square$

Lemma 4.3. *The support of ν is contained in the closure of the set $V_{t,\nu}^{-1}$, which is defined by*

$$V_{t,\nu}^{-1} = \left\{ \frac{1}{x} : x \in V_{t,\nu} \right\}.$$

Proof. Let $x_0 \in (0,+\infty)\backslash\overline{V_{t,\nu}}$, and choose $\epsilon > 0$ sufficiently small such that $[x_0-\epsilon, x_0+\epsilon] \subset (0,+\infty)\backslash\overline{V_{t,\nu}}$, it is enough to show that ν has no charge on the interval $[1/(x_0+\epsilon), 1/(x_0-\epsilon)]$. For any integer $n \geq 1$, we define $\beta_k = x_0 - \epsilon + 2k\epsilon/n$ for all $0 \leq k \leq n$. Then since $[\beta_k, \beta_{k+1}] \subset (0,+\infty)\backslash\overline{V_{t,\nu}}$, then for any $r \in [\beta_k, \beta_{k+1}]$, we have that

$$\begin{aligned} \frac{1}{t} &\geq \int_{1/\beta_{k+1}}^{1/\beta_k} \frac{r\xi}{(1-r\xi)^2} d\nu(\xi) \\ &\geq \frac{1}{(x_0+\epsilon)^2 m_k} \nu([1/\beta_{k+1}, 1/\beta_k]), \end{aligned}$$

where $m_k = \max\{|1-\beta_k/\beta_{k+1}|^2, |1-\beta_{k+1}/\beta_k|^2\}$. This yields that

$$\nu([1/\beta_{k+1}, 1/\beta_k]) \leq \frac{(x_0+\epsilon)^2}{t} m_k.$$

Notice that $\sum_{k=0}^{n-1} m_k = o(1/n)$ for large n, thus we have $\nu([1/(x_0+\epsilon), 1/(x_0-\epsilon)]) = 0$. $\square$

Proposition 4.4. *For any interval $(a,b) \subset (0,+\infty)\backslash\overline{V_{t,\nu}}$, the function $g(r) = \int_0^{+\infty} \frac{r\xi}{(1-r\xi)^2} d\nu(\xi)$ is strictly convex on (a,b).*

Proof. We first note that ν has no charge on the interval $(1/b, 1/a)$ by Lemma 4.3, thus g is analytic on the interval (a, b). The second derivative of g is

$$g''(r) = \int_0^{+\infty} \frac{4\xi^2 + 2r\xi^2}{(1-r\xi)^4} d\nu(\xi) > 0,$$

which yields the desired conclusion. $\square$

Proposition 4.4 yields that $\int_0^{+\infty} \frac{r\xi}{(1-r\xi)^2} d\nu(\xi) < 1/t$ for r in the interior of $(0, +\infty)\backslash V_{t,\nu}$. We now define a map $\Lambda_{t,\nu} : (0, +\infty) \to (0, +\infty)$ by

$$\Lambda_{t,\nu}(r) = H_{t,\nu}(re^{iu_t(r)}).$$

Note that $\arg(H_{t,\nu}(re^{iu_t(r)})) = 0$, we then have

$$\Lambda_{t,\nu}(r) = r \exp\left(\frac{t}{2}\int_0^{+\infty} \frac{r^2\xi^2 - 1}{|1 - r\xi e^{iu_t(r)}|^2} d\nu(\xi)\right).$$

Proposition 4.5. *The map $r \to re^{iu_t(r)}$ is a homeomorphism from $(0, +\infty)$ onto $\partial\Gamma_{t,\nu}\backslash(-\infty, 0]$; and the map $\Lambda_{t,\nu}$ is a homeomorphism from $(0, +\infty)$ onto itself.*

Define the Cauchy transform of the probability measure ν by

$$G_\nu(z) = \int_{-\infty}^{+\infty} \frac{1}{z-t} d\nu(t),\ z \in \mathbb{C}\backslash\mathbb{R},$$

we then have

$$G_\nu\left(\frac{1}{z}\right) = \frac{z}{1 - \eta_\nu(z)}, \tag{4.2}$$

thus the density of the measures can be recovered by (4.2) from the Stieltjes inverse formula. We notice that $1/(1 - \eta_\nu(z)) = 1 + \psi_\nu(z)$, which implies that

$$\begin{aligned}\Im\left(\frac{1}{1-\eta_\nu(z)}\right) &= \Im\left(\int_0^{+\infty} \frac{1}{1-z\xi} d\nu(\xi)\right) \\ &= \int_0^{+\infty} \frac{r\xi\sin\theta}{1 + r^2\xi^2 - 2r\xi\cos\theta} d\nu(\xi),\end{aligned}$$

where $z = re^{i\theta}$. From Remark 4, for $z = re^{iu_t(r)}$, we have that

$$\Im\left(\frac{1}{1 - \eta_\nu\left(re^{iu_t(r)}\right)}\right) = \frac{u_t(r)}{t}. \tag{4.3}$$

Equations (4.2) and (4.3), together with the Stieltjes inverse formula yield the following theorem.

Theorem 4.6. *The measure ν_t has a density given by*

$$q_t\left(\frac{1}{\Lambda_{t,\nu}(r)}\right) = \Lambda_{t,\nu}(r)\frac{1}{\pi}\frac{u_t(r)}{t}$$

for $r \in (0, +\infty)$.

Proof. It is a direct calculation.

$$\begin{aligned}
q_t\left(\frac{1}{\Lambda_{t,\nu}(r)}\right) &= q_t\left(\frac{1}{H_{t,\nu}(re^{iu_t(r)})}\right)\\
&= \frac{1}{\pi}\Im\left(G_{\nu_t}\left(\frac{1}{H_{t,\nu}(re^{iu_t(r)})}\right)\right)\\
&= \frac{1}{\pi}H_{t,\nu}(re^{iu_t(r)})\Im\left(\frac{1}{1-\eta_{\nu_t}(H_{t,\nu}(re^{iu_t(r)}))}\right)\\
&= \frac{1}{\pi}H_{t,\nu}(re^{iu_t(r)})\Im\left(\frac{1}{1-\eta_\nu\left(re^{iu_t(r)}\right)}\right)\\
&= \Lambda_{t,\nu}(r)\frac{1}{\pi}\frac{u_t(r)}{t}.
\end{aligned}$$

□

Corollary 4.7. *Let B_t be the support of the measure ν_t, and let $B_t^{-1} = \{1/x : x \in B_t\}$, then B_t^{-1} equals to the image of the closure of $V_{t,\nu}$ by the homeomorphism $\Lambda_{t,\nu}$. The number of connected components of the interior of the support of ν_t is a non-increasing function of t.*

Proof. Theorem 4.6 and the fact that

$$u_t(r) \neq 0 \Leftrightarrow r \in V_{t,\nu}$$

yield the first assertion. To prove the second assertion, it is enough to show that the number of connected components of $V_{t,\nu}$ is a non-increasing function of t, which follows from Proposition 4.4. □

The following lemma was pointed out in [2].

Lemma 4.8 ([2])**.** *Given a probability measure μ on $[0,+\infty)$, the point $x = 0$ is an atom if and only if $\lim_{x\downarrow-\infty}\eta_\mu(x)$ is finite and the value of the limit is*

$$\lim_{x\downarrow-\infty}\eta_\mu(x) = 1 - \frac{1}{\mu(\{0\})}.$$

Proposition 4.9. *The measure ν_t is absolutely continuous with respect to Lebesgue measure and its density q_t is analytic on the set $\{x \in (0,+\infty) : q_t(x) > 0\}$.*

Proof. Since $\eta_{\sigma_t}(\overline{\mathbb{C}^+})$ does not contain 1 (see Example 4.11), and η_{ν_t} is subordinated with respect to η_{σ_t}, then $\eta_{\nu_t}(\mathbb{C}^+\backslash\{0\})$ does contain 1 as well. From (4.2), we then see that the nontangential limit of G_{ν_t} is bounded at any $x \in (0,+\infty)$, which implies that ν_t has no singular part in $(0,+\infty)$.

If the density of ν_t is positive at $1/x$, then from (4.2), we deduce that $\eta_{\nu_t}(x) \in \mathbb{C}^+$. Thus the identity

$$\frac{z}{\omega_t(z)} = \exp\left(\frac{t}{2}\frac{\eta_{\nu_t}(z)+1}{\eta_{\nu_t}(z)-1}\right) \tag{4.4}$$

implies that $\omega_t(x) \in \mathbb{C}^+$, and then the analyticity of the density function follows from the fact that ω_t can be extended analytically to a neighborhood of x.

It remains to show that $\nu_t(\{0\}) = 0$. Notice that $H_{t,\nu}(\omega_t(x)) = x$ for $x \in (-\infty, 0)$ and $\lim_{x\downarrow-\infty} H_{t,\nu}(x) = -\infty$, we then have that $\lim_{x\downarrow-\infty} \omega_t(x) = -\infty$. Thus we have that

$$\lim_{x\downarrow-\infty} \frac{x}{\omega_t(x)} = \lim_{x\downarrow-\infty} \Sigma_{\tau_t}(\omega_t(x)) = \exp\left(\frac{t}{2}\right),$$

which also implies that $\lim_{x\downarrow-\infty} \eta_{\nu_t}(x) = -\infty$ by (4.4). From Lemma 4.8, we deduce that $\nu_t(\{0\}) = 0$. $\square$

Corollary 4.10. *If ν is compactly supported on $(0, +\infty)$, then the interior of the support of ν_t has only one connected component for large t.*

Proof. We show that $V_{t,\nu}$ has only one connected component when t is large enough. By Proposition 4.4, the function g is continuous on $(0, +\infty)\backslash V_{t,\nu}$, then the assertion follows from the fact that g has a positive minimum on any compact subset of $(0, +\infty)\backslash V_{t,\nu}$ and the definition of $V_{t,\nu}$. $\square$

Finally, we apply results in this section to give a description of the density of σ_t, which was first obtained in [8]. See also [11, 15] for different approaches. We provide a picture of the density function of σ_t in the end of this paper.

Example 4.11. Let $\nu = \delta_1$, we denote $\Gamma_t = \Gamma_{t,\delta_1}, \Lambda_t = \Lambda_{t,\delta_1}$ and $V_t = V_{t,\delta_1}$, we also set $H_t(z) = H_{t,\delta_1}(z) = z\exp\left(\frac{t}{2}\frac{z+1}{z-1}\right)$. Then we have that $\nu_t = \sigma_t$ and

$$V_t = \left\{0 < r < +\infty : \frac{r}{(1-r)^2} > \frac{1}{t}\right\} = (x_1(t), x_2(t)), \tag{4.5}$$

where $x_1(t) = (2 + t - \sqrt{t(t+4)})/2$ and $x_2(t) = (2 + t + \sqrt{t(t+4)})/2$. If $r \in V_t$, then $u_t(r)$ satisfies the equation

$$\frac{\sin(u_t(r))}{u_t(r)} \frac{r}{1 + r^2 - 2r\cos(u_t(r))} = \frac{1}{t}, \tag{4.6}$$

and we have

$$\frac{r\sin(u_t(r))}{1 + r^2 - 2r\cos(u_t(r))} = \Im\left(\frac{1}{1 - re^{iu_t(r)}}\right). \tag{4.7}$$

We also have

$$\Lambda_t(r) = r\exp\left(\frac{t}{2}\frac{r^2 - 1}{|1 - re^{iu_t(r)}|^2}\right),$$

and in particular, when $u_t(r) = 0$, we have $\Lambda_t(r) = r\exp\left(\frac{t}{2}\frac{r+1}{r-1}\right)$. We calculate

$$x_3(t) := \frac{1}{\Lambda_t(x_2(t))} = \frac{2 + t - \sqrt{t(t+4)}}{2}\exp\left(-\frac{\sqrt{t(t+4)}}{2}\right).$$

and

$$x_4(t) := \frac{1}{\Lambda_t(x_1(t))} = \frac{2 + t + \sqrt{t(t+4)}}{2}\exp\left(\frac{\sqrt{t(t+4)}}{2}\right),$$

where we used the fact that $u_t(x_1(t)) = u_t(x_2(t)) = 0$. We remark that $x_1(t)x_2(t) = x_3(t)x_4(t) = 1$.

Note that the formulas we use are different from the formulas used in [8] by Biane, we then record our description of the density of σ_t as follows.

Proposition 4.12. *The support of the measure σ_t is equal to the interval $C_t = [x_3(t), x_4(t)]$. The density is positive on the interior of the interval C_t, and is equal to, at the point $x \in C_t$, to $(1/(\pi x))\Im(l(t,x))$, where $l(t,x)$ is the unique solution of the equation*

$$\frac{z}{z-1}\exp\left(t\left(z-\frac{1}{2}\right)\right) = x$$

on Γ_t.

Proof. By the definition of $x_3(t), x_4(t)$ and Theorem 4.6, we see that $C_t = [x_3(t), x_4(t)]$. Let $r = \Lambda_t^{-1}(1/x)$, then by Theorem 4.6, the density at the point x is equal to $(1/(\pi x))u_t(r)/t$. By (4.6) and (4.7),

$$\frac{u_t(r)}{t} = \Im\left(\frac{1}{1-re^{iu_t(r)}}\right), \tag{4.8}$$

we then set $l(t,x) = 1/(1-re^{iu_t(r)})$. Since $\Lambda_t(r) = 1/x$, we then have that $H_t(re^{iu_t(r)}) = 1/x$. The description of $l(t,x)$ follows from this identity and the definition of H_t. □

We now turn to discuss symmetry of the measure σ_t.

Proposition 4.13. *For $\nu = \delta_1$, u_t is a strictly increasing function of r on the interval $(x_1(t), 1]$, u_t is a strictly decreasing function of r on the interval $[1, x_2(t))$, and $u_t(r) = 0$ for all $r \in (0,+\infty)\backslash(x_1(t), x_2(t))$. In particular, u_t attains its global maximum at 1.*

Proof. Let g be the function defined by

$$g(r,\theta) = \frac{\sin\theta}{\theta}\frac{r}{1+r^2-2r\cos\theta}$$

on the set $(0,+\infty)\times[0,\pi)$. The first part of the assertion follows from (4.6) and the following fact:

(i) $\frac{\partial g(r,\theta)}{\partial r} < 0$, for $(r,\theta) \in [1,+\infty)\times[0,\pi)$.
(ii) $\frac{\partial g(r,\theta)}{\partial r} > 0$, for $(r,\theta) \in (0,1]\times[0,\pi)$.
(iii) $\frac{\partial g(r,\theta)}{\partial \theta} < 0$, for $(r,\theta) \in (0,+\infty)\times(0,\pi)$.

The rest part follows from (4.5). □

Lemma 4.14. *For $\nu = \delta_1$, we have*

$$u_t(r) = u_t\left(\frac{1}{r}\right), \text{ and } \Lambda_t(r)\cdot\Lambda_t\left(\frac{1}{r}\right) = 1,$$

holds for all $r > 0$.

Proof. We first note that $r \in V_t = (x_1(t), x_2(t))$ if and only if $1/r \in V_t$. If $r \notin V_t$, then $u_t(r) = 0$, and we have

$$\Lambda_t(r) = r\exp\left(\frac{t}{2}\frac{r+1}{r-1}\right) \text{ and } \Lambda_t\left(\frac{1}{r}\right) = \frac{1}{r}\exp\left(\frac{t}{2}\frac{r+1}{1-r}\right),$$

which prove the case when $r \notin V_t$.

If $r \in V_t$, we first prove that $u_t(r) = u_t(1/r)$. Recall from (4.6) that the pair $(r, u_t(r))$ satisfies the equation

$$\frac{\sin\theta}{\theta}\frac{r}{1+r^2-2r\cos\theta} = \frac{1}{t}$$

which is equivalent to

$$r^2 - \left(2\cos\theta + t\frac{\sin\theta}{\theta}\right) r + 1 = 0. \tag{4.9}$$

For any θ, (4.9) is a quadratic equation of r and the product of its solutions is 1. This observation and Proposition 4.13 implies that $u_t(r) = u_t(1/r)$. For any $\theta \in (0, u_t(1)]$, let r_1, r_2 be the solutions of (4.9), then $r_1, r_2 \in (0, +\infty)$ and $r_1 \cdot r_2 = 1$. We claim that $\Lambda_t(r_1) \cdot \Lambda_t(r_2) = 1$. If fact, $\theta = u_t(r_1) = u_t(r_2)$, and

$$\ln\left[\Lambda_t(r_i)\right] = \ln r_i + \frac{t}{2}\frac{r_i^2 - 1}{1 + r_i^2 - 2r_i\cos\theta}, \text{ for } r = 1, 2. \tag{4.10}$$

By replacing

$$1 + r_i^2 - 2r_i\cos\theta = \frac{tr_i\sin\theta}{\theta}$$

into (4.10), to prove our claim, it is enough to prove that

$$r_2r_1^2 - r_2 + r_1r_2^2 - r_1 = 0. \tag{4.11}$$

Since $r_1r_2 = 1$, (4.11) is true and this finishes the proof. □

Proposition 4.15. *For $t > 0$, let $s_t(x)$ be the density of the measure σ_t at $x \in (0, +\infty)$.*

(1) *The support of the function s_t is the interval $[x_3(t), x_4(t)]$. For any $x \in (0, +\infty)$, we have that*

$$s_t(x)x = s_t\left(\frac{1}{x}\right)\frac{1}{x}.$$

Moreover, the function $x \to s_t(x)x$ is strictly increasing on the interval $(x_3(t), 1]$.

(2) *For any interval $[\alpha, \beta] \subset (0, 1]$, we have that*

$$\sigma_t([\alpha, \beta]) = \sigma_t\left(\left[\frac{1}{\beta}, \frac{1}{\alpha}\right]\right).$$

(3) *The function s_t is a strictly decreasing function of x on the interval $[1, x_4(t))$.*

Proof. From Theorem 4.6 and Lemma 4.14, we have that

$$s_t\left(\frac{1}{\Lambda_t(r)}\right) = \Lambda_t(r)\frac{1}{\pi}\frac{u_t(r)}{t}, \tag{4.12}$$

and that

$$\begin{aligned} s_t(\Lambda_t(r)) &= s_t\left(\frac{1}{\Lambda_t\left(\frac{1}{r}\right)}\right) \\ &= \frac{1}{\pi}\Lambda_t\left(\frac{1}{r}\right)\frac{u_t\left(\frac{1}{r}\right)}{r} \\ &= \frac{1}{\pi}\frac{1}{\Lambda_t(r)}\frac{u_t(r)}{t}. \end{aligned} \tag{4.13}$$

Note that $\Lambda_t : (0, +\infty) \to (0, +\infty)$ is a homeomorphism and Λ_t is increasing. We also have that $\Lambda_t(V_t) = \Lambda_t((x_1(t), x_2(t))) = (x_3(t), x_4(t))$, then (1) follows from Proposition 4.13, Lemma 4.14 and (4.13). Part (2) is a consequence of (1). Since u_t is an increasing function of r on the interval $(x_1(t), 1]$ by Proposition 4.13, then $\Lambda_t u_t$ is also an increasing function of r on $(x_1(t), 1]$ and Part (3) follows from this fact and (4.12). $\square$

Remark 5. Part (2) of Proposition 4.15 is not unexpected. In fact, for any integer n, let $a_n = 1 + \sqrt{2t/n}$, $b_n = 1/a_n$ and $\mu_n = (\delta_{a_n} + \delta_{b_n})/2$, then from [4, Lemma 7.1], we see that $\mu_n^{\boxtimes n} \to \sigma_t$ weakly as $n \to +\infty$.

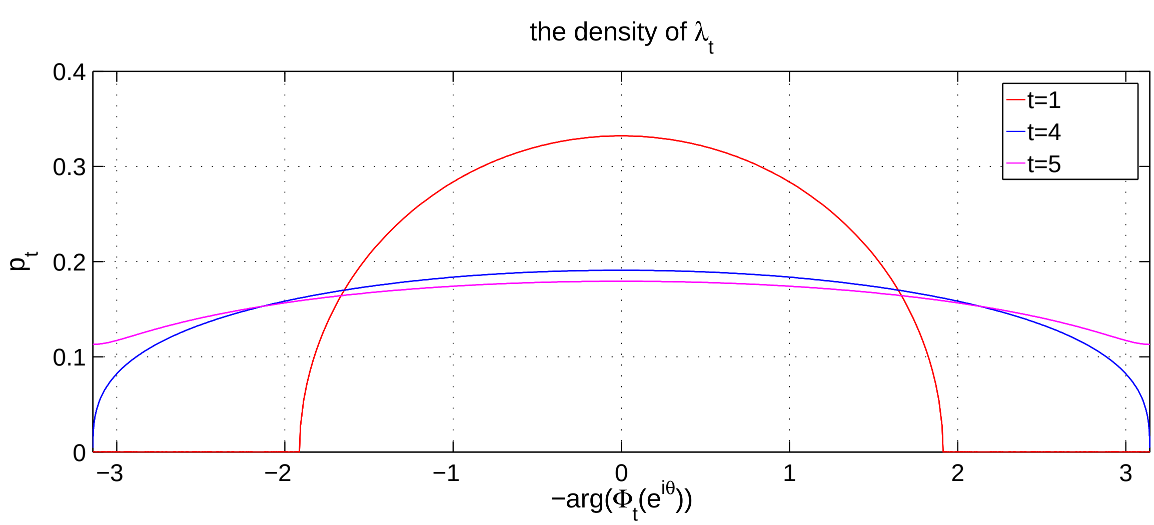
Remark 6. By a similar calculation in Remark 3, we can see that there exist $c(t) > 0, d(t) > 0$ such that

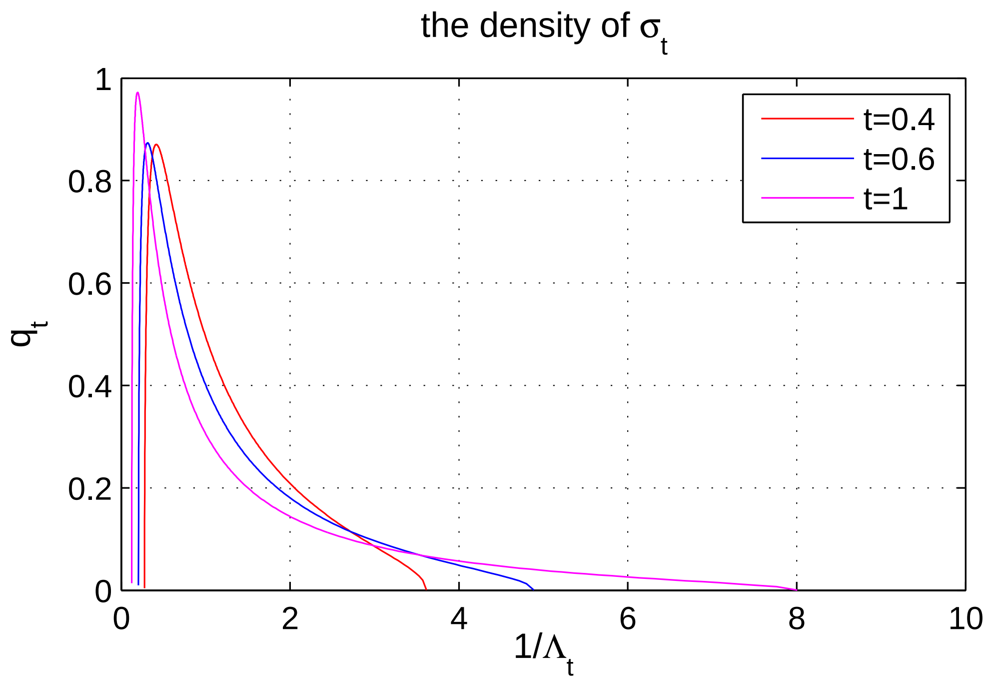
$$s_t(x) = c(t)|x - x_3(t)|^{\frac{1}{2}}(1 + o(1)) \tag{4.14}$$

in a small interval $[x_3(t), x_3(t) + \delta)$ for some $\delta > 0$, and that

$$s_t(x) = d(t)|x - x_4(t)|^{\frac{1}{2}}(1 + o(1)) \tag{4.15}$$

in a small interval $(x_4(t) - \delta', x_4(t)]$ for some $\delta' > 0$. The orders in (4.14) and (4.15) are essentially due to the fact that H_t has zeros of order one at $x_1(t)$ and $x_2(t)$.





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Department of Mathematics, Rawles Hall, 831 East Third Street, Indiana University, Bloomington, Indiana 47405, U.S.A.

E-mail address: `pzhong@indiana.edu`